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ON THE APPLICATION OF THE METHOD OF SYMBOLIC INTEGRATION IN THE THEORY OF PIEZOCERAMIC SHELLS*

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A variant is proposed of the theory of piezoceramic shells, with various directions of polarization and means of electric loading, based on the method of symbolic integration /1/.

The Kirchhoff-Love hypotheses with additional assumptions about the variation of electric field components along the thickness of the shell was used in /2,3/ for calculating thin piezoceramic shells polarized along the normal to the shell median surface. Equations of motion were obtained in /4/ for a shell with the same polarization direction on the basis of the variational principle of electroelasticity and the quadratic approximation of the electric field potential along the thickness coordinate. Shells of revolution with meridional polarization and various conditions of electrical loading were considered in /5/, where the "electrical" correction to the Kirchhoff-Love hypotheses was introduced, which allowed a considerable simplification of the solution of the problem in the case of an arbitrary shell of revolution.

1. Consider a piezoceramic shell of thickness $2h$ whose faces are either completely covered by electrodes or are free of them. The equations of equilibrium obtainable with the Kirchhoff-Love hypotheses are of the form /6/

$$L_1\{T_1,\ T_2,\ S,\ M_1,\ M_2,\ H\} = -A_1A_2q_1,\ L_2\{\ldots\} = -A_1A_2q_2,\ L_3\{\ldots\} = q_n \tag{1.1}$$

Using the Kirchhoff-Love hypotheses we write the equations of state for the piezoceramic shell with the polarization thickness in the form

$$\sigma_{11} = c_{11}{}^*\varepsilon_{11} + c_{12}{}^*\varepsilon_{22} - e_{31}{}^*E_3,\quad \sigma_{12} = \frac{c_{11}{}^E - c_{12}{}^E}{2}\varepsilon_{12} \tag{1.2}$$

$$\sigma_{22} = c_{12}{}^*\varepsilon_{11} + c_{11}{}^*\varepsilon_{22} - e_{31}{}^*E_3$$

$$D_3 = \varepsilon_{33}{}^*E_3 + e_{31}{}^*(\varepsilon_{11} + \varepsilon_{22}),\quad D_i = \varepsilon_{11}{}^SE_i\quad (i = 1,\ 2) \tag{1.3}$$

Taking into account (1.2) we represent the relation between stresses, moments, deformations, and the electric field potential in the form

$$T_{1,2} = 2c_{11}{}^*h(\varepsilon_{1,2} + \nu_*\varepsilon_{2,1}) + e_{31}{}^*T_0,\quad S = (c_{11}{}^E - c_{12}{}^E)\,h\omega \tag{1.4}$$

$$M_{1,2} = \frac{2c_{11}{}^*h^3}{3}(\varkappa_{1,2} + \nu_*\varkappa_{2,1}) + e_{31}{}^*M_0,\quad H = \frac{(c_{11}{}^E - c_{12}{}^E)\,h^3}{3}\tau$$

The quantities T_0 and M_0 introduced in (1.4) are determined by the relations

$$T_0 = \varphi^+ - \varphi^-,\quad M_0 = h(\varphi^+ + \varphi^-) - \int_{-h}^{h}\varphi\,dz \tag{1.5}$$

where φ is the electric field potential, and $\varphi^\pm$ are its values at $z = \pm h$.

Taking into account the relation of $\varepsilon_1, \varepsilon_2, \omega$ and $\varkappa_1, \varkappa_2, \tau$ with displacements of the median shell surface, and the Maxwell equations for piezoceramic body /2,3,8/

$$\operatorname{div}\mathbf{D} = 0,\ \operatorname{rot}\mathbf{E} = 0 \tag{1.6}$$

we obtain from Eqs.(1.1) and (1.3)-(1.6) a closed system of equations in the potential and the components of the displacement vector of the median surface of the shell. In view of the complexity of solution of the obtained system of equations, we use the method of symbolic integration /1/ widely applied in problems of thermo-elasticity of thin shells /7/. In conformity with that method we represent the first of Eqs.(1.6) with allowance for (1.3) and for the second of Eqs.(1.6)

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$$\frac{d^2\varphi}{dz^2} + p^2\varphi = \frac{e_{31}^*}{\varepsilon_{33}^*}(\varkappa_1 + \varkappa_2) \tag{1.7}$$

$$p^2 = \frac{\varepsilon_{11}{}^S}{\varepsilon_{33}^*}\frac{1}{A_1A_2}\left[\frac{\partial}{\partial\alpha_1}\frac{A_2}{A_1}\frac{\partial(\ldots)}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2}\frac{A_1}{A_2}\frac{\partial(\ldots)}{\partial\alpha_2}\right]$$

We introduce the integral characteristics of the electric field potential by the relations

$$\Phi_i = \int\limits_{-h}^{h} z^{i-1}\varphi(\alpha_1, \alpha_2, z)\,dz \quad (i = 1, 2) \tag{1.8}$$

Writing the general solutions of (1.7) in the form

$$\varphi = \cos pz C_1(\alpha_1, \alpha_2) + \sin pz C_2(\alpha_1, \alpha_2) + \frac{e_{31}^*(\varkappa_1 + \varkappa_2)}{\varepsilon_{33}^* p^2}$$

we obtain in conformity with (1.8) Φ_1 and Φ_2, after which the field potential is represented by the formula

$$\varphi = \frac{p\cos pz}{2\sin ph}\Phi_1 + \frac{p^2\sin pz}{2\sin ph\,(1 - ph\operatorname{ctg} ph)}\Phi_2 - \frac{e_{31}^* h\cos pz}{\varepsilon_{33}^* p\sin ph}(\varkappa_1 + \varkappa_2) + \frac{e_{31}^*(\varkappa_1 + \varkappa_2)}{\varepsilon_{33}^* p^2} \tag{1.9}$$

Let us consider two basic cases of loading.

1°. The face surfaces of the shell are covered with electrodes and the value of potential $\varphi^{\pm} = \pm V_0$ is specified on them. Then from (1.5) we have

$$T_0 = 2V_0, \quad M_0 = -\Phi_1$$

Satisfying condition $\varphi(\alpha_1, \alpha_2, \pm h) = V_0$, from (1.9) we have the equations for determining Φ_1 and Φ_2. Retaining in them terms up to $(ph)^2$, we obtain the respective approximate equations

$$\left(1 - \frac{p^2h^2}{3}\right)\Phi_1 = -\frac{2e_{31}^* h^3(\varkappa_1 + \varkappa_2)}{3\varepsilon_{33}^*}, \quad \left(1 - \frac{p^2h^2}{15}\right)\Phi_2 = \frac{2V_0h^2}{3}$$

After the determination of Φ_1, Φ_2, we calculate the field potential using the approximate formula

$$\varphi = \left(1 + \frac{h^2 - 3z^2}{6}p^2\right)\frac{\Phi_1}{2h} + \frac{3z}{2h^3}\left(1 + \frac{3h^2 - 5z^2}{30}p^2\right)\Phi_2 - \frac{e_{31}^*(h^2 - 3z^2)}{6\varepsilon_{33}^*}(\varkappa_1 + \varkappa_2) \tag{1.10}$$

obtained from (1.9) on the assumption that the electric field potential varies over the shell thickness in conformity with the cube law. If, however, one makes the assumption that the field potential distribution over the shell thickness is quadratic, then we have

$$\varphi = \left(1 + \frac{h^2 - 3z^2}{6}p^2\right)\frac{\Phi_1}{2h} + \frac{3z\Phi_2}{2h^3} - \frac{e_{31}^*(h^2 - 3z^2)}{6\varepsilon_{33}^*}(\varkappa_1 + \varkappa_2) \tag{1.11}$$

In that case we have

$$\Phi_2 = {}^2/_3 V_0 h^2 \tag{1.12}$$

and the equation for Φ_1 remains unchanged.

The boundary conditions, specified on the shell contour, consist of the usual mechanical conditions and the conditions for the potential

$$R\varphi = -\frac{D}{\varepsilon_{11}{}^S}, \quad R = \frac{\cos\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \frac{\sin\gamma}{A_2}\frac{\partial}{\partial\alpha_2} \tag{1.13}$$

where γ is the angle between the normal to the contour of the median surface and the coordinate line α_1 and D is the electric induction.

In conformity with (1.8) from (1.13) we obtain

$$R\Phi_i = -\Phi_i^*, \quad \Phi_i^* = \frac{1}{\varepsilon_{11}{}^S}\int\limits_{-h}^{h} z^{i-1}D\,dz \quad (i = 1,2)$$

When the field potential is distributed in accordance with (1.11), the boundary condition for Φ_2 is not specified.

2°. Let the face surfaces $z = \pm h$ of the shell be free of electrodes, and on them the normal component of electrical induction vector is specified

$$D_3(\alpha_1, \alpha_2, \pm h) = D^{\pm} \tag{1.14}$$

Using the first of relations (1.3) and satisfying conditions (1.14) we obtain the equations for Φ_1, Φ_2. Retaining in the expansion of operators appearing in these equations the terms up to $(ph)^2$, we obtain the approximate equations

$$p^2\Phi_1 = \frac{-(D^+ - D^-)}{\varepsilon_{33}^*}, \quad \left(1 - \frac{2p^2h^2}{5}\right)\Phi_2 = \frac{(D^+ + D^-)h^3}{3\varepsilon_{33}^*} - \frac{2e_{31}^*h^3(\varepsilon_1 + \varepsilon_2)}{\varepsilon_{33}^*} \tag{1.15}$$

Note that the first of Eqs. (1.15) is exact.

Assuming that the field potential is distributed over the shell thickness in accordance with (1.10) law, then for T_0 and M_0 with (1.9), and (1.15) taken into account, we obtain the respective approximate formulas

$$T_0 = \frac{3}{h^2}\left(1 - \frac{p^2h^2}{15}\right)\Phi_2, \quad M_0 = \frac{(D^+ + D^-)h^2}{3\varepsilon_{33}^*} - \frac{2e_{31}^*h^3(\varkappa_1 + \varkappa_2)}{3\varepsilon_{33}^*} \tag{1.16}$$

If the field potential is distributed over the shell thickness according to (1.11), we can approximately consider that

$$\Phi_2 = \frac{(D^+ + D^-)h^3}{3\varepsilon_{33}^*} - \frac{2e_{31}^*h^3(\varepsilon_1 + \varepsilon_2)}{3\varepsilon_{33}^*}$$

$$T_0 = \frac{(D^+ + D^-)h}{\varepsilon_{33}^*} - \frac{2e_{31}^*h(\varepsilon_1 + \varepsilon_2)}{\varepsilon_{33}^*}$$

The expressions for M_0 and Φ_1 remain unchanged.

Assuming that on the part of the shell side surface S_1 the potential V_1 is specified, while on part $S_2 = S_0/S_1$ (S_0 is the side surface of the shell) the condition of the form (1.13) is given, we obtain

$$\begin{aligned} &\Phi_1 = 2V_1h, \quad \Phi_2 = 0 \quad \text{on } S_1 \\ &R\Phi_i = -\Phi_i^* \quad (i = 1, 2) \quad \text{on } S_2 \end{aligned} \tag{1.17}$$

If the permittivity of the external medium is considerably smaller than the permittivity of ceramics, for instance in the case of air, we can assume that $D^+ = D^- = 0$. Otherwise D^+, D^- are unknown quantities whose determination requires the consideration of the equations of the electrostatics the external medium.

2. Consider a piezoceramic shell of thickness $2h$ polarized along one of the coordinate lines (for definiteness, for example, along α_1). Obviously Eqs. (1.1) and (1.6) hold for a shell with such polarization direction. From the equations of state for ceramics polarized along the coordinate line α_1, taking into account the Kirchhoff-Love hypotheses, we obtain

$$\begin{aligned} &\sigma_{11} = c_{33}^*\varepsilon_{11} + c_{13}^*\varepsilon_{22} - e_{33}^*E_1, \quad \sigma_{12} = c_{44}^E\varepsilon_{12} - e_{15}E_2 \\ &\sigma_{22} = c_{13}^*\varepsilon_{11} + g_{11}^*\varepsilon_{22} - \vartheta_{31}^*E_1 \\ &D_3 = \varepsilon_{11}^SE_3, \quad D_2 = e_{15}\varepsilon_{12} + \varepsilon_{11}^SE_2, \quad D_1 = \beta_{33}^*E_1 + \vartheta_{31}^*\varepsilon_{22} + e_{33}^*\varepsilon_{11} \end{aligned} \tag{2.1}$$

where the following notation has been used:

$$c_{33}^* = c_{33}^E\left(1 - \frac{(c_{13}^E)^2}{c_{33}^Ec_{11}^E}\right), \quad c_{13}^* = c_{13}^E(1 - \nu^E), \quad \nu^E = \frac{c_{12}^E}{c_{11}^E}$$

$$g_{11}^* = c_{11}^E(1 - (\nu^E)^2), \quad \vartheta_{31}^* = e_{31}(1 - \nu^E)$$

$$\beta_{33}^* = \varepsilon_{33}^S\left(1 + \frac{e_{31}^2}{c_{11}^E\varepsilon_{33}^S}\right), \quad e_{33}^* = e_{33}\left(1 - \frac{e_{31}c_{13}^E}{e_{33}c_{11}^E}\right)$$

The formulas for stresses and moments are of the form

$$\begin{aligned} &T_1 = 2h(c_{33}^*\varepsilon_1 + c_{13}^*\varepsilon_2) + e_{33}^*\frac{1}{A_1}\frac{\partial\Phi_1}{\partial\alpha_1} \\ &T_2 = 2h(c_{13}^*\varepsilon_1 + g_{11}^*\varepsilon_2) + \vartheta_{31}^*\frac{1}{A_1}\frac{\partial\Phi_1}{\partial\alpha_1} \end{aligned} \tag{2.2}$$

$$S = 2c_{44}{}^E h\omega + e_{15}\frac{1}{A_2}\frac{\partial\Phi_1}{\partial\alpha_2}$$
$$M_1 = \frac{2h^3}{3}(c_{33}{}^*\varkappa_1 + c_{13}{}^*\varkappa_2) + e_{33}{}^*\frac{1}{A_1}\frac{\partial\Phi_2}{\partial\alpha_1}$$
$$M_2 = \frac{2h^3}{3}(c_{13}{}^*\varkappa_1 + g_{11}{}^*\varkappa_2) + \partial_{31}{}^*\frac{1}{A_1}\frac{\partial\Phi_2}{\partial\alpha_1}$$
$$H = \frac{2c_{44}{}^E h^3}{3}\tau + e_{15}\frac{1}{A_2}\frac{\partial\Phi_2}{\partial\alpha_2}$$

where Φ_1, Φ_2 are defined by relations (1.8).

Let us write the first of Eqs. (1.6). Taking into account the last three relations (2.1),

$$\frac{d^2\varphi}{\partial z^2} + q^2\varphi = f_1 + zf_2 \tag{2.3}$$
$$q^2 = \left[\frac{\beta_{33}{}^*}{\varepsilon_{11}{}^S}\frac{\partial}{\partial\alpha_1}\frac{A_2}{A_1}\frac{\partial(\ldots)}{\partial\alpha_1} + \frac{\partial}{\partial\alpha_2}\frac{A_1}{A_2}\frac{\partial(\ldots)}{\partial\alpha_2}\right]\frac{1}{A_1A_2}$$
$$f_1 = \frac{1}{\varepsilon_{11}{}^S}\left\{\frac{\partial}{\partial\alpha_1}A_2(\partial_{31}{}^*\varepsilon_2 + e_{33}\varepsilon_1) + e_{15}\frac{\partial}{\partial\alpha_2}A_1\omega\right\}\frac{1}{A_1A_2}$$
$$f_2 = \frac{1}{\varepsilon_{11}{}^S}\left\{\frac{\partial}{\partial\alpha_1}A_2(\partial_{31}{}^*\varkappa_2 + e_{33}{}^*\varkappa_1) + e_{15}\frac{\partial}{\partial\alpha_2}A_1\tau\right\}\frac{1}{A_1A_2}$$

The electric field potential is expressed in terms of integral characteristics Φ_1 and Φ_2

$$\varphi = \frac{q\cos qz}{2\sin qh}\Phi_1 + \frac{q^2\sin qz}{2\sin qh\,(1 - qh\operatorname{ctg} qh)}\Phi_2 - \frac{\cos qz}{q\sin qh}f_1 + \frac{f_1}{q^2} - \frac{h^3\sin qz}{3\sin qh\,(1 - qh\operatorname{ctg} qh)}f_2 + z\frac{f_2}{q^2} \tag{2.4}$$

Omitting the obvious transformations, we present the final equations for the determination of Φ_1 and Φ_2 and, also, indicate the necessary boundary conditions corresponding to 1° and 2°.

1°. Assuming the potential of field to vary over the shell thickness in conformity with the cube law

$$\varphi = \left(1 + \frac{h^2 - 3z^2}{6}q^2\right)\frac{\Phi_1}{2h} + \frac{3z}{2h^3}\left(1 + \frac{3h^2 - 5z^2}{30}q^2\right)\Phi_2 - \frac{h^2 - 3z^2}{6}f_1 - z\frac{3h^2 - 5z^2}{30}f_2$$

we obtain the approximate equations for Φ_1 and Φ_2

$$\left(1 - \frac{q^2h^2}{3}\right)\Phi_1 = \frac{2h^3f_1}{3}, \quad \left(1 - \frac{q^2h^2}{15}\right)\Phi_2 = \frac{2V_0h^3}{3} - \frac{2h^5f_2}{45} \tag{2.5}$$

If the field potential varies according to the law

$$\varphi = \left(1 + \frac{h^2 - 3z^2}{6}q^2\right)\frac{\Phi_1}{2h} + \frac{3z}{2h^3}\Phi_2 - \frac{h^2 - 3z^2}{6}f_1 \tag{2.6}$$

then for Φ_1 the first of Eqs. (2.5) is valid, and Φ_2 is determined by the formula (1.12). From the condition

$$\cos\gamma\, D_1 + \sin\gamma\, D_2 = D \quad \text{on} \quad S_0$$

taking into account the last two relations (2.1), we obtain the boundary conditions for Φ_1 and Φ_2, which finally assume the form

$$R_1\Phi_1 = -\Phi_1{}^* + 2h\{\cos\gamma\,(\partial_{31}{}^*\varepsilon_2 + e_{33}{}^*\varepsilon_1) + e_{15}\sin\gamma\,\omega\} \tag{2.7}$$
$$R_1\Phi_2 = -\Phi_2{}^* + \frac{2h^3}{3}\{\cos\gamma\,(\partial_{31}{}^*\varkappa_2 + e_{33}\varkappa_1) + e_{15}\sin\gamma\tau\}$$
$$R_1 = \beta_{33}{}^*\frac{\cos\gamma}{A_1}\frac{\partial}{\partial\alpha_1} + \varepsilon_{11}{}^S\frac{\sin\gamma}{A_2}\frac{\partial}{\partial\alpha_2}$$

If however the field potential varies according to (2.6), the second condition in (2.7) is omitted.

2°. The approximate equations for the determination of Φ_1 and Φ_2, obtained from condition (1.14), are of the form

$$q^2\Phi_1 = \frac{D^+ - D^-}{\varepsilon_{11}{}^S} - hf_1, \quad \left(1 - \frac{2q^2h^2}{5}\right)\Phi_2 = \frac{(D^+ + D^-)h^3}{3\varepsilon_{11}{}^S} - \frac{4h^5f_2}{15}$$

The boundary conditions on portion S_1, where potential V_1 is given, of the form (1.17) and on portion S_2, where the normal component of induction vector is given are of the form (2.7).

3. Consider the problem of stable oscillations of a rectangular piezoceramic plate. Let a and b be linear dimensions along the x and y axes, respectively. The plate is polarized over the thickness $2h$, the face surfaces are without electrodes, and on them the constant value of normal component of electrical induction vector that harmonically varies is time. Let us assume that the plate performs only bending oscillations, and that $D^+ - D^- = Q\exp(i\Omega t), D^+ + D^- = 0$. In conformity with formulas (1.1), (1.4), (1.15) and (1.16) with allowance for the identity

$$\frac{(\nu_* + k_*^2)c_{11}{}^* + c_{11}{}^E(1 - \nu^E)}{c_{11}{}^*(1 + k_*^2)} = 1$$

for the determination of deflection $w^* = w/a$ and the integral characteristics of the electric field potential we obtain

$$\nabla_*\nabla_* w^* = \varkappa^2\delta^{-2}w^*, \ \nabla_* G_1 = 1, \ \Phi_2 = 0 \tag{3.1}$$

$$\nabla_* = \frac{\partial^2}{\partial x_*^2} + \frac{\partial^2}{\partial y_*^2}, \quad x_* = \frac{x}{a}, \quad y_* = \frac{y}{a}, \quad G_1^* = \frac{\Phi_1}{f_3}$$

$$f_3 = \frac{Qa^2}{\varepsilon_{33}{}^*}, \quad \delta^2 = \frac{h^2}{3a^2}, \quad \varkappa^2 = \frac{\Omega^2 a^2}{c_{11}{}^*(1 + k_*^2)}, \quad k_*^2 = \frac{(e_{31}{}^*)^2}{c_{11}{}^*\varepsilon_{33}{}^*}$$

The mechanical boundary conditions that correspond to a hinged edge are of the form

$$w^* = 0, \ \frac{\partial^2 w^*}{\partial x_*^2} + \frac{\nu_* + k_*^2}{1 + k_*^2}\frac{\partial^2 w^*}{\partial y_*^2} = Q^\circ, \quad x_* = \pm 1 \tag{3.2}$$

$$w^* = 0, \ \frac{\nu_* + k_*^2}{1 + k_*^2}\frac{\partial^2 w^*}{\partial x_*^2} + \frac{\partial^2 w^*}{\partial y_*^2} = Q^\circ, \quad y_* = \pm\frac{b}{a}$$

$$Q^\circ = \frac{e_{31}{}^* aQ}{2c_{11}{}^*\varepsilon_{33}{}^*(1 + k_*^2)h}$$

Assuming that the contour of the plate is covered by electrodes and the value of its potential is zero, we write the boundary condition on the contour for G_1

$$G_1 = 0 \tag{3.3}$$

It follows from relations (3.1)– (3.3) that the problem of forced oscillations of the piezoceramic plate under the action of induction difference, reduces to the problem of forced oscillations of an isotropic plate under the action of the specified (on the contour bending) moment of the quantity Q°. In the absence of the piezoeffect for the assumed frequency we have $f^2 = \rho\Omega^2 a^2/c_{11}{}^*$. Hence the allowance for the piezoeffect results in a hightening of the resonant frequency by a factor of $(1 + k_*^2)$. For piezoceramic $PZT = 4$ [8] $k_*^2 = 0.3233$. The electric field potential, after solving Eq.(3.1) and satisfying the condition (3.3) is determined by formula (1.11).

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